Convective Instabilities in Concurrent Two-Phase Flow:

RAYMOND J. GUMERMAN and GEORGE M. HOMSY

Deptrtment of Chemical Engineering
Stanford University
Stanford, California 94305

Part II. Global Stability

The method of energy is used to determine global stability limits for thermally stratified two-phase plane Couette flow. Instabilities due to surface tension variations, buoyancy effects, and shear are allowed for, but surface waves are specifically excluded from consideration. Under these assumptions, the Marangoni, Rayleigh, and Reynolds numbers completely describe the system. Stability plots, valid for disturbances of any magnitude, are presented for both streamwise oriented roll vortices and two-dimensional transversely oriented disturbances. It is shown that rolls are the most dangerous disturbances in the sense that they cannot be shown to be stable relative to transverse disturbances at any nonzero Reynolds numbers. Comparisons are made with existing linear limits, and these are seen to be close only for moderate Reynolds numbers.

SCOPE

This study is the second in a series which focuses on convective instabilities due to surface tension, density, or velocity gradients in two phase flow. The need for a unified stability theory for these phenomena is clear: in the cases in which they occur, they significantly affect the rates of heat, mass, or momentum transport between the phases. These increases have yet to be exploited in any systematic manner in applications for a variety of reasons. The foremost of these pertains to the relative uncertainty in being able to predict when instabilities will and will not occur. As we have noted (Gumerman and Homsy, 1974), most theoretical analyses have been for stagnant plane layers, while most practical applications involve the presence of shear, for example, liquid-liquid extraction and film evaporation. In that work, we employed the techniques of linear theory to discuss the various modes of instability and gave quantitative stability limits for these modes.

Linear stability studies of this sort give sufficient conditions for instability. For stability parameters (Reynolds number, Rayleigh number, etc.) above the critical values, instabilities are guaranteed. In this paper, we develop limits for global stability of a stratified flow without regard for the magnitude of disturbances. This procedure yields sufficient conditions for absolute stability: below the global limits the flow and corresponding temperature or concentration fields are unique.

Linear and global stability theories are thus seen to be complementary, and stability theory is of great utility when these limits are close. In this paper we give global limits for a slightly restricted class of disturbances, compare the results with available linear limits, and note the conditions under which these limits are close.

CONCLUSIONS AND SIGNIFICANCE

The model system employed here is that of thermally stratified plane Couette flow. The model is specifically restricted to fluid pairs with large density differences and large interfacial tension. These restrictions exclude the existence of interfacial waves discussed extensively in Part I (Gumerman and Homsy, 1974). Thus, instabilities due to surface tension, density, and velocity gradients are allowed for.

It is shown that of the possible disturbance modes, streamwise oriented roll vortices are the most dangerous, that is, they yield the lowest global stability boundaries. These results are valid for disturbances of arbitrary magnitude. For the longitudinal roll mode, the linear limits are independent of Reynolds number and are given by Nield's (1964) calculations. A comparison of the global and linear limits (Figure 1) shows them to be close only for low Reynolds numbers.

Results are also given for disturbances oriented transverse to the flow (Figure 2). There are only limited linear limits in this case, so no detailed comparison may be made. The results substantiate those in Part I in indicating that rolls may be expected for fluid pairs of unequal densities. Lastly, we note that the results in Figures 1 and 2 are expected to be relatively insensitive to the details of the base flow and hence should be qualitatively the same for other two-phase flows. There is thus a large region of parameter space for which subcritical finite amplitude disturbances may be expected in concurrent two-phase flow.

Correspondence concerning this paper should be addressed to G. M. Homsy. R. J. Gumerman is with Chevon Research Corporation, Richmond, California.

The energy method gives the means of determining the absolute stability of hydrodynamic systems with respect to disturbances of arbitrary amplitude. Its result is a prediction of the critical value of some system parameter, such as Reynolds, Rayleigh, or Marangoni number, below which stability is guaranteed. When the base state (the undisturbed system) is stable, all disturbances will be found to exponentially decay with time. Consequently the base state is a unique solution to the governing equations in this case. We note that the energy method will also give an upper bound to the decay rate in the stable state.

The energy method obviously complements the well-known small disturbance technique of linear hydrodynamic stability theory. This latter technique gives sufficient conditions for the instability of systems—the value of a parameter is found above which ubiquitous small amplitude disturbances will grow. Consequently, for regions of parameter space between the predictions of the two stability methods, the system may be unstable to large amplitude disturbances, a so-called "subcritical instability." Stability theory clearly gives its most useful results when the two limits are close. Fortunately this is the case for a number of important problems. In particular, Joseph (1966) showed that the energy theory gives the same result as small disturbance theory for the Bénard problem. An interesting resulting prediction is that a pool of fluid undergoing buoyancy driven convection will stagnate (when cooled) at the same Rayleigh number that will cause instability in a heated stagnant pool. This is of course verified experimentally. In addition, the predictions of the two stability theories are close in the surface tension driven instability problem (Davis, 1969). Results are typically less useful in systems susceptible to parallel shear flow instability. For as yet undetermined reasons, small disturbance theory gives a critical Reynolds number far in excess of that observed experimentally, which in turn greatly exceeds that of the energy method (Joseph and Carmi, 1969).

We now discuss the general formulation of the energy method. The basic approach is to study the time behavior of some modulus of the difference motion. The difference motion is simply the difference between the perturbed flow and the base state whose stability we are studying. In contrast to linear theory, the energy method retains terms which are nonlinear in the difference motion. Hence, results of the energy method are equally valid for both large and small amplitude disturbances. At conditions for which the modulus can be shown to decay, the system can be proven to be globally stable.

In convective instability problems the modulus is defined as $E = 1/2 < \vec{u} \cdot \vec{u} + \lambda\theta^2 >$ where $\vec{u}$ and θ are the velocity and temperature fields of the difference motion and the brackets indicate integration over the body of the fluid. λ is an arbitrary nonnegative coupling parameter between the two fields. From the dynamic equations for the difference motion we develop a relation between energy dissipating terms $\mathcal{D}$ and energy production terms I_λ:

$$\frac{dE}{dt} = R\, I_\lambda - \mathcal{D} \tag{1.1}$$

Here R is the Rayleigh or Marangoni number and $\mathcal{D}$ is a generalized dissipation function which is positive definite. The form of I_λ depends upon the problem under discussion, and explicit relations will be given below. It is clear from Equation (1.1) that for sufficiently large R, the energy will increase (if I_λ is positive), and the flow is unstable. Conversely, there exists an R below which the energy will always decrease. The prediction of R, and hence the global stability limit of the flow, is the aim of the energy stability method.

Equation (1.1) is rearranged to yield

$$\frac{dE}{dt} = R\, \mathcal{D}(I_\lambda/\mathcal{D} - 1/R) \tag{1.2}$$

Since $\mathcal{D}$ is positive definite, it is clear from (1.2) that the difference motion decays for $R < \rho_\lambda$ where

$$\frac{1}{\rho_\lambda} = \max_{h} \frac{I_\lambda(\vec{u}, \theta)}{\mathcal{D}(\vec{u}, \theta)} \tag{1.3}$$

To get the best stability limit it is clear that the function space h should be restricted to dynamically admissible solutions $(\vec{u}, \theta)$; however, it is mathematically impossible to solve (1.3) if we restrict the space so severely. Typically the variational calculus problem of (1.3) is solved by restricting $\vec{u}$ to be kinematically admissible and θ to satisfy certain boundary conditions. In addition, it is usual to add the differential constraint of conservation of mass to the function space h. Finally, the optimal stability limit is given by adjusting the disposable parameter λ in such a way as to give the largest region of stability. This is clearly given by $R \leq \tilde{\rho} \equiv \max_{\lambda>0} \rho_\lambda$.

It is possible to show exponential decay below the predicted stability limit using the isoperimetric inequality,

$$\mathcal{D} \geq \xi^2 E \tag{1.4}$$

where ξ^2 is positive and depends upon the geometry (Joseph, 1965). Hence

$$\frac{dE}{dt} \leq E\, \xi^2 \left(\frac{R}{\tilde{\rho}} - 1\right) \quad \text{whenever} \quad R < \tilde{\rho}, \tag{1.5}$$

and

$$E(t) \leq E(0)\, e^{t\, \xi^2 \left[\frac{R}{\tilde{\rho}} - 1\right]} \tag{1.6}$$

Thus the modulus of the disturbance motion decreases exponentially in time if the flow is globally stable.

It is worthwhlie at this point to examine more closely previous work on related problems, especially Joseph (1966) and Davis (1969), because the present work is a combination and extension of their work. In addition to applying the energy method to stagnant pools, Joseph considered buoyancy instabilities in plane Couette flow. His results indicate that longitudinally oriented disturbances are more dangerous than are transverse disturbances. By *more dangerous,* we mean that these disturbances give the lowest $\tilde{\rho}$, and hence these are the ones which cannot be shown to decay for stability parameters R which are greater than $\tilde{\rho}$. Increasing Reynolds number decreased the critical Rayleigh number, as expressed in the following relationship for the stability boundary:

$$Re^2 + Ra = R_L$$

where R_L is the critical Rayleigh number for the stagnant problem. The linear stability theory for this problem was developed by Gallagher and Mercer (1965), Deardorff (1965), and Ingersoll (1966). These works show small amplitude longitudinal rolls to be more unstable than transverse disturbances. However, since Couette flow is stable by linear theory, linear theory applied to this problem gives a shear independent stability boundary.

Davis (1969) considered buoyancy and surface tension instabilities in a horizontal heated layer with a nondeformable upper surface and a rigid conducting lower surface. When the upper surface thermal condition is insulating, subcritical instabilities are confined to a narrow band. For other thermal conditions results are not as strong, but in all cases the band of allowable subcritical instabilities vanishes at small Marangoni numbers, as required by Joseph's results.

The objective of the present work is to apply the energy theory to a system analogous to that studied in Part I (Gumerman and Homsy, 1974). As discussed in Part I, there is little theoretical work concerning the effect of shear upon convective instabilities, whereas the problem is of great practical importance. The results in Part I showed that, at moderate Reynolds numbers, the two competing instability modes are streamwise oriented roll vortices and long interfacial waves. Explicit formulae giving the critical parameters for onset of interfacial waves were given therein. The corresponding parameters for rolls may be obtained from known results for plane layers (Smith, 1966; Nield, 1964; Zeren and Reynolds, 1971). For fluid pairs with high density ratios and sufficiently high interfacial tension, Nield's (1964) model yields fairly accurate predictions of the stability parameters for onset of convection.

In this paper we restrict ourselves to cases in which the interfacial wave discussed in Part I is strongly damped, and for which Nield's one-fluid model is a good approximation to the more involved, exact two-fluid calculations of Zeren and Reynolds. In this way we may apply the method of energy to this problem in order to (1) assess the effect of shear on the orientation of the most dangerous mode and (2) present a demarcation of the regions of uniqueness in parameter space. We therefore neglect the surface deformation of the free interface. By additionally neglecting the viscous effects in the adjoining phase, we have a system characterized by only three dimensionless groups: the Marangoni number, the Rayleigh number, and the Reynolds number. While the restriction to planar interfaces eliminates the traveling transverse mode discussed extensively in Part I, the present work will give two important results: First, it will indicate that longitudinal vortices are the most dangerous mode when the interface is constrained to be flat. Secondly, by comparison with Nield (1964), it will give the region in parameter space where subcritical instabilities are possible. Nield's linear instability results for simultaneous buoyancy and surface driven convection in a stagnant pool are also limits for any plane-parallel flow, since as we have shown in Part I, the two effects do not interact in the linear regime.

Lastly, we note that, in contrast to linear instability predictions, the results of the energy method are relatively insensitive to the details of the velocity profile; see, for example, Joseph and Carmi (1969). We therefore expect that the results and conclusions developed here for plane Couette flow will not differ greatly for more general flow profiles of interest in stratified two-phase flows.

BASE STATE AND STABILITY FORMULATION

We consider a horizontal liquid layer of thickness d heated from below. The layer is bounded below by an isothermal rigid surface located at $z = 0$. The upper nondeformable surface is moving with a steady velocity V_1 in the y direction and is subject to surface tension variations. To allow for buoyancy instability, the Boussinesq approximation is made. Using this model we note that Davis' (1969) results will be recovered at zero Reynolds number, while Joseph's (1966) results will be recovered at zero Marangoni number.

The equations governing the base state and the difference motion will be made dimensionless using the scalings

$$\{\vec{r}, t, \vec{u}, V, p, \theta, T\} = \left\{ d, d^2/\kappa, \kappa/d, V_1, \frac{\rho\nu\kappa}{d^2}, \Delta T, \Delta T \right\} \tag{2.1}$$

The base state has a steady linear velocity profile we will call $V(z)$. The base state temperature profile is also linear since the surface temperatures are held at constant values.

The dimensionless nonlinear disturbance equations are

$$\frac{1}{\sigma}\frac{\partial \vec{u}}{\partial t} + \frac{1}{\sigma}\vec{u}\cdot\nabla\vec{u} + Re\, V(z)\frac{\partial \vec{u}}{\partial y} + Re\, w\frac{\partial V}{\partial z}\hat{j} = -\nabla p + \nabla^2\vec{u} + Ra\,\theta\,\hat{k} \tag{2.2a}$$

$$\frac{\partial \theta}{\partial t} + \vec{u}\cdot\nabla\theta + Re\,\sigma\, V(z)\frac{\partial\theta}{\partial y} + w\frac{\partial T}{\partial z} = \nabla^2\theta \tag{2.2b}$$

On the lower surface the boundary conditions are

$$\theta = \vec{u} = 0 \tag{2.2c}$$

while on the upper surface, $z = 1$, they are

$$\frac{\partial v}{\partial z} + Ma\frac{\partial\theta}{\partial y} = \frac{\partial u}{\partial z} + Ma\frac{\partial \theta}{\partial x} = 0 \tag{2.2d}$$

$$\frac{\partial\theta}{\partial z} + L\theta = 0 \tag{2.2e}$$

$$w = 0 \tag{2.2f}$$

Equations (2.2d) show the balance of viscous stresses and surface tension traction at the free surface. Equation (2.2e) is a mixed thermal condition: a so-called "insulating condition" is recovered if $L = 0$, while a conducting interface is represented by $L = \infty$. The last limit is unreasonable in this problem—no surface instability can occur (by 2.2d) if the upper surface is isothermal. Thus Davis finds no Marangoni instability is possible in this limit.

An energy identity, similar to (1.1), is obtained by taking the scalar product of $\vec{u}$ with (2.2a) and the product of $\lambda\theta$ with (2.2b), adding the two equations, and integrating the result over the entire volume of the fluid:

$$\frac{dE}{dt} = -Re\left\langle vw\frac{\partial V}{\partial z}\right\rangle + (Ra + \lambda)\langle\theta w\rangle - Ma\int_{z=1}\theta\frac{\partial w}{\partial z} - D(\vec{u},\vec{u}) - \lambda\mathfrak{D}(\theta,\theta) + \langle\nabla\cdot\vec{\psi}\rangle \tag{2.3a}$$

Here E is the energy functional,

$$E = \frac{1}{2}\langle\vec{u}\cdot\vec{u}/\sigma + \lambda\theta^2\rangle \tag{2.3b}$$

and

$$D(\vec{u},\vec{u}) = \langle\nabla\vec{u} : \nabla\vec{u}\rangle$$

$$\mathfrak{D}(\theta,\theta) = \langle\nabla\theta\cdot\nabla\theta\rangle + L\int_{z=1}\theta^2 \tag{2.3c}$$

$$\vec{\psi} = p\,\vec{u} + Re\frac{1}{2}V\hat{j}(\vec{u}\cdot\vec{u}) + \frac{1}{2}\sigma^{-1}\vec{u}(\vec{u}\cdot\vec{u}) + \lambda Re\,\sigma\frac{1}{2}\hat{j}V\theta^2 + \lambda\theta^2\vec{u}/2 \tag{2.3d}$$

The term $\langle\nabla\cdot\vec{\psi}\rangle$ in Equation (2.3a) is zero. This follows by using the divergence theorem and noting that $w = 0$ at $z = 0$, and that the disturbance u, θ is periodic (by assumption) in the horizontal directions. The only nonzero surface integral in (2.3a) is the term

$$-Ma\int_{z=1}\theta\frac{\partial w}{\partial z}.$$

It is interesting to see how the nonzero terms of Equation (2.3) influence the disturbance energy. The first term is a positive Reynolds stress term that increases disturbance energy: it is positive for reasons discussed in Schlicting (1968). The second term represents the positive input caused by the destabilizing effect of buoyancy. The third term represents the destabilizing effect of surface tension forces. It is positive, since liquid approaching the interface is characterized by $\theta > 0$, $\partial w/\partial z < 0$, whereas fluid moving away from the interface has $\theta < 0$, $\partial w/\partial z > 0$. The last two terms in (2.3a) tend to decrease the disturbance energy—they represent the effects of viscous dissipation and thermal conduction, respectively.

In order to get the problem into convenient form, we make the following transformation:

$$Re = \mu_1 Ma^{1/2}, \quad Ra = \mu_2 Ma, \quad \lambda = Ma\,\eta, \quad \theta = \varphi/\lambda^{1/2}$$

We have thus parameterized Re, Ra and scaled the temperature disturbances. Using these substitutions transforms Equation (2.3a) to

$$\frac{dE}{dt} = -\mu_1 Ma^{1/2} \left\langle vw \frac{\partial V}{\partial z} \right\rangle + Ma^{1/2} \frac{\mu_2 + \eta}{\eta^{1/2}} \langle \varphi w \rangle - \frac{Ma^{1/2}}{\eta^{1/2}} \int_{z=1} \varphi \frac{\partial w}{\partial z} - D(\vec{u}, \vec{u}) - \mathcal{D}(\varphi, \varphi)$$

which is of the same form as (1.1) with $R = Ma^{1/2}$. It is then clear that for this problem [see (1.3)]

$$\frac{1}{\rho_\eta} = \max_h \left\{ -\mu_1 \left\langle vw \frac{\partial V}{\partial z} \right\rangle + \frac{\mu_2 + \eta}{\sqrt{\eta}} \langle \varphi w \rangle - \frac{1}{\sqrt{\eta}} \int_{z=1} \varphi \frac{\partial w}{\partial z} \right\} \quad (2.4a)$$

The function space h of couples $(\vec{u}, \varphi)$ satisfies the normalizing condition, $D(\vec{u}, \vec{u}) + \mathcal{D}(\varphi, \varphi) = 1$; the constraint of incompressibility $\nabla \cdot \vec{u} = 0$, and the essential boundary conditions,

$$\frac{\partial w}{\partial z}(0) = w(0) = \varphi(0) = w(1) = 0 \quad (2.4b)$$

Note that $w(0) = \frac{\partial w}{\partial z}(0) = 0$ is equivalent to $\vec{u}(0) = \vec{0}$.

Instead of solving Equations (2.4) by direct techniques, it is simpler to work with the Euler-Lagrange equations of the variational problem. These are readily found to be

$$\nabla \cdot \vec{u} = 0 \quad (2.5a)$$

$$-\frac{2}{\rho_\eta} \nabla p + \frac{\mu_2 + \eta}{\sqrt{\eta}} \varphi \hat{k} - \mu_1 wDV \hat{k} = -\frac{2}{\rho_\eta} \nabla^2 \vec{u} \quad (2.5b)$$

$$\frac{\mu_2 + \eta}{\sqrt{\eta}} w = -\frac{2}{\rho_\eta} \nabla^2 \varphi \quad (2.5c)$$

with the natural boundary conditions (at $z = 1$)

$$\frac{\rho_\lambda}{2\sqrt{\eta}} Dw + D\varphi + L\varphi = 0 \quad (2.5d)$$

$$\frac{\rho_\lambda}{2\sqrt{\eta}} \frac{\partial \varphi}{\partial y} + Dv = \frac{\rho_\lambda}{2\sqrt{\eta}} \frac{\partial \varphi}{\partial x} + Du = 0 \quad (2.5e)$$

$p(x, y, z)$ is a Lagrange multiplier associated with the solenoidal constraint on $\vec{u}$.

SOLUTION

Equations (2.5) pose an eigenvalue problem with a set of eigenvalues $\{\rho_{\eta,i}\}$. The smallest of these, here called ρ_η, is obviously the eigenvalue of interest (see 2.4). While the coupling parameter η is to be adjusted to yield the best stability limit (that is, the largest ρ), there appears to be no way to determine its value analytically as done in Joseph (1966). It must be sought in the numerical solution of the problem.

It is convenient to remove the (x, y) dependence of w and φ by a Fourier decomposition. We assume that elements of the function space h are sufficiently smooth and differentiable that φ and w can be represented as the Fourier transfrom of a function. Hence

$$\begin{Bmatrix} w(x, y, z, t) \\ \varphi(x, y, z, t) \end{Bmatrix} = \iint_{-\infty}^{\infty} e^{i(K_x x + K_y y)} \begin{Bmatrix} \hat{w} \\ \hat{\varphi} \end{Bmatrix} dK_x\, dK_y.$$

In addition to the wave numbers K_x and K_y, a total wave number $K = (K_x^2 + K_y^2)^{1/2}$ appears. In fact, K is the only wave number to appear at $Re = 0$. In the equations to follow, the circumflex over φ and w is omitted.

Equations (2.5a) and (2.5b) can be combined to give the useful results

$$(D^2 - K^2)^3 w + K_y \mu_1 \rho_\eta i (D^2 - K^2) Dw + \left\{ K^2 \left(\frac{\mu_2 + \eta}{\sqrt{\eta}} \right)^2 + K_x^2 \mu_1^2 \right\} \left(\frac{\rho_\eta}{2} \right)^2 w = 0 \quad (2.6)$$

$$K^2 p = (D^2 - K^2) Dw + iK_y \mu_1 \frac{\rho_\eta}{2} w \quad (2.7)$$

Because of the Reynolds stress terms of Equations (2.5b), it was not convenient to eliminate φ in favor of w. Since w appears in linear equations with constant complex coefficients, it is expressed as a sum of exponentials. From w, φ can be obtained by solving Equation (2.5c). Thus

$$w = \sum_{i=1}^{6} A_i e^{im_i z} \quad (2.8a)$$

$$\varphi = \sum_{i=1}^{6} A_i \frac{\rho_\lambda}{2} \frac{\mu_2 + \eta}{\sqrt{\eta}} e^{im_i z} + A_7 e^{Kz} + A_8 e^{-Kz} \quad (2.8b)$$

One boundary condition comes from taking the z derivative of the $\hat{k}$ component of (2.5b) using (2.7) to eliminate ρ and applying the result at $z = 1$ where (2.5e) is used to eliminate v:

$$(\mu_2 + \eta) D\varphi + iK_y \frac{\rho_\lambda}{2} \mu_1 \varphi + \frac{2\sqrt{\eta}}{\rho_\lambda} \left\{ (D^2 - K^2) Dw - \frac{1}{K^2} (D^2 - K^2) D^3 w - \frac{iK_y \mu_1}{K^2} \frac{\rho}{2} D^2 w \right\} = 0 \quad (2.9)$$

Equations (2.5e) are combined to give

$$D^2 w + \frac{\rho_\lambda}{2} K^2 \frac{1}{\sqrt{\eta}} \varphi = 0 \quad (2.10)$$

The $\hat{k}$ component of (2.5b) and (2.7) are combined at $z = 0$ to give the usual restatement of the isothermal condition at a rigid boundary

$$(D^2 - K^2)^2 w = 0 \quad (2.11)$$

Equations (2.9), (2.10), (2.11), and (2.4b) are the eight equations required to determine the unknowns A_i of Equations (2.8).

We now outline the numerical method for solving the eigenvalue problem. η was transformed to λ/ρ_λ^2 during the calculations. At fixed $\{K_x/K_y, Ra, Re, \lambda, K, \text{and } L\}$, ρ_λ was determined by a direct search. The roots $\{m_i\}$ were found by substituting (2.8a) into (2.6) and solving for the roots of the polynomial using a standard subroutine. The $\{m_i\}$ at the estimated ρ_λ were then used to check to see if the absolute value of the determinant of the boundary coefficient matrix vanished. New estimates of ρ_λ were found using a Golden section search. This procedure was repeated until the smallest eigenvalue ρ_λ was located.

The energy stability boundary was then located through the following saddlepoint problem:

$$Ma^{1/2} = \max_{\lambda} \min_{K_x,K_y} \rho_\lambda \tag{2.12}$$

It is necessary to minimize over the wave number to ensure stability of the most dangerous Fourier mode.

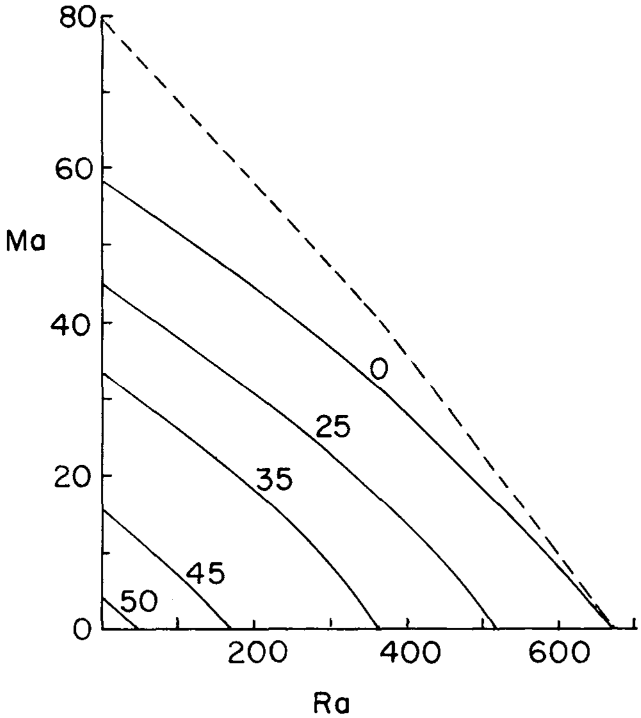

Fig. 1. Stability results for longitudinal rolls at $L = 0$. Re is the parameter. — — = Nield's linear instability theory for all Re.

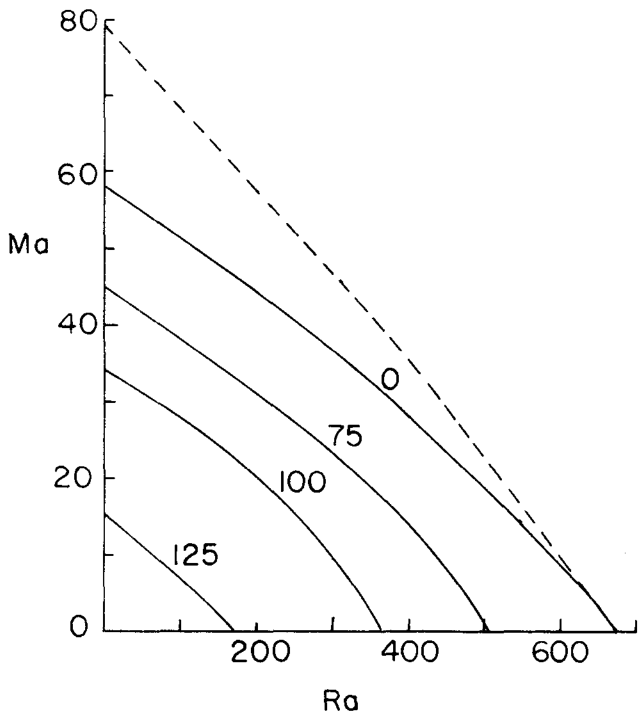

Fig. 2. Stability results for transverse disturbances at $L = 0$. Re is the parameter. — — — = Nield's linear instability theory for $Re = 0$.

RESULTS

The results of the parametric study are presented in Figures 1 and 2 for longitudinal and transverse disturbances, respectively. Stability is guaranteed for parameter values below the indicated lines, whereas disturbances may grow at conditions above them.

The present results check well with previous work. For both the transverse and longitudinal cases, Davis' (1969) results are recovered in the limit of $Re = 0$. Furthermore, Joseph's (1966) results for longitudinal rolls are recovered when $Ma \to 0$.

As we noted in the introduction, a unified stability theory is most useful when linear and global limits are available and lie close to each other. Recall, linear theory gives sufficient conditions for instability; energy theory gives sufficient conditions for absolute stability. In the present case, we can display linear limits for rolls, but only limited linear limits for waves. As we pointed out in 1, the linear limits for rolls are unaffected by the flow: thus for this mode Nield's (1964) results are the instability limits for all Reynolds numbers. The corresponding calculations for transverse waves have yet to be reported. For $Re = 0$, we of course recover Nield's results. On the other hand, for homogeneous fluids ($Ra = Ma = 0$) the indications are that Couette flow is linearly stable. Thus, the linear stability picture is far from complete in this case.

Several important observations can be made on the basis of these results. First, it is clear that longitudinal vortices are the most dangerous mode of convection whenever shear is present and the interface is flat regardless of the relative importance of buoyancy and surface effects. This statement can be made since the results for longitudinal vortices fall far below those for the transverse case, indicating greater stability of transverse disturbances.

Second, it is clear that shear decreases the global stability of convectively unstable systems. While the energy method is generally conservative in predictions of parallel shear flow instability, these results at small Reynolds number where convective instabilities are the dominant mode are probably reasonable. This statement is justified by the fact that the linear stability results of Nield (1964) and energy results of Davis (1969) are very close. There is, however, a large region of parameter space at moderate Re where subcritical finite amplitude instabilities may be expected.

It should be pointed out that the results of Figure 1 are not the best possible; that is, the stability bounds can be raised. Joseph and Hung (1971) have recently shown that the absence of a disturbance pressure gradient in one direction allows the uncoupling of the evolution equation for $u \cdot u$. For longitudinal rolls there exist separate energy equations for v^2, and $u^2 + w^2$. Another coupling parameter analogous to λ can be introduced in this case. The value of this parameter is implicitly equated to unity in the development above. Optimizing ρ with respect to this second parameter would result in some improvement in the stability boundary, which we briefly discuss in the Appendix.

In summary, we have used the energy method to determine regions of absolute stability of plane Couette flow. The results shown in Figure 1 give the regions in parameter space for which the flow is globally asymptotically stable. The results shown in Figure 2 give regions of stability for two-dimensional wavelike disturbances of arbitrary amplitude. These results are expected to be relatively insensitive to the exact form of the velocity profile.

NOTATION

A_i = the solution vector of Equation (2.8)
d = thickness of liquid layer, m
D = (1) viscous dissipation, dimensionless
(2) derivative with respect to z, dimensionless
$\mathfrak{D}$ = (1) a general dissipation term in the introduction, dimensionless
(2) otherwise, a conductive dissipation term, dimensionless
E = disturbance energy modulus, dimensionless
h = a function space
I_λ = general energy production term in Introduction, dimensionless
$\hat{j}$ = unit vector along y axis, dimensionless
K_x = wave number in x-direction, dimensionless
K_y = wave number in y-direction, dimensionless
K = $(K_x^2 + K_y^2)^{1/2}$
$\hat{k}$ = unit vector along z axis, dimensionless
L = Biot number, dimensionless
Ma = Marangoni number
m_i = roots defined in Equation (2.8)
p = disturbance pressure; Lagrange multiplier in Equation (2.5) *et seq.*, dimensionless
$\vec{r}$ = position vector, m
R = a general characteristic parameter in the Introduction
Ra = Rayleigh number
Re = Reynolds number
t = time, s
T = base state temperature, K
ΔT = temperature difference across the layer, K
$\vec{u}$ = disturbance velocity vector, dimensionless
v = disturbance velocity in y direction, dimensionless
V = base state velocity in y direction, dimensionless
V_1 = $V(z = 1)$ (m/s)
w = disturbance velocity in z direction, dimensionless
x = cross stream coordinate, dimensionless
y = coordinate parallel to the base flow, dimensionless
z = coordinate in the vertical direction, dimensionless
∇ = the divergence operator
∇^2 = the Laplacian operator
η = rescaled coupling constant, $= \lambda/Ma$
θ = the disturbance temperature, dimensionless
κ = thermal diffusivity, m²/s
λ = a coupling parameter
μ_1 = a parameter, $= Re/Ma^{1/2}$
μ_2 = a parameter, $= Ra/Ma$
ν = kinematic viscosity, m²/s
ξ = a geometry dependent constant in the isoperimetric inequality
ρ = density, kg/m³
ρ_λ = the minimum of the functional in (2.4a) or (1.3)
$\tilde{\rho}$ = the maximum of ρ_λ with respect to λ
σ = Prandtl number
φ = rescaled temperature field $= \theta\lambda^{1/2}$, dimensionless
$\vec{\psi}$ = vector defined by (2.3d), dimensionless

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APPENDIX

For the transverse disturbance case, the disturbance velocity in the x-direction, u, vanishes. There is no force to drive this velocity component. Hence, there is no additional coupling parameter associated with v, w.

In the longitudinal case, a Reynolds stress term drives v and we can associate a coupling parameter ω with u, w. The problem formulation is the same as before except that we multiply the $\hat{i}$ and $\hat{k}$ parts of Equation (2.2a) by ω before further operation. Equation (2.3a) becomes

$$\frac{dE}{dt} = - Re \left\langle vw \frac{\partial V}{\partial z} \right\rangle - <\nabla v \cdot \nabla v>$$
$$- \omega <\nabla u \cdot \nabla u + \nabla w \cdot \nabla w> + Ra <w\theta>$$
$$= Ma\, \omega \int_{z=1} \theta \frac{\partial w}{\partial z} + \lambda <\theta w> - <\nabla\theta \cdot \nabla\theta> \quad \text{(A1)}$$

where

$$E = \left\langle v^2/\sigma + \frac{\omega}{\sigma}(u^2 + w^2) + \lambda\theta^2 \right\rangle \quad \text{(A2)}$$

Rescaling u and w using $u' = \sqrt{\omega}\, u$ and $w' = \sqrt{\omega}\, w$ gives

$$\frac{dE}{dt} = - <\nabla v \cdot \nabla v> - <\nabla u' \cdot \nabla u'$$
$$+ \nabla w' \cdot \nabla w'> - <\nabla\varphi \cdot \nabla\varphi>$$
$$+ \sqrt{Ma} \left\{ - \frac{\mu_1}{\sqrt{\omega}} \left\langle vw' \frac{\partial V}{\partial z} \right\rangle - \sqrt{\frac{\omega}{\eta}} \int_{z=1} \varphi \frac{\partial w'}{\partial z} + \frac{(\mu_2 + \eta)}{\sqrt{\omega\eta}} <\varphi w'> \right\} \quad \text{(A3)}$$

From (A3) it is clear that the maximum problem is very similar to (2.4a)

$$\rho(\eta, \omega) = \max_h \left\{ - \frac{\mu_1}{\sqrt{\omega}} \left\langle vw' \frac{\partial V}{\partial z} \right\rangle - \sqrt{\frac{\omega}{\eta}} \int \varphi \frac{\partial w'}{\partial z} + \frac{\mu_2 + \eta}{\sqrt{\eta\omega}} <\varphi w'> \right\} \quad \text{(A4)}$$

For this maximum problem the Euler Lagrange equations are easily found to be

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad \text{(A5)}$$

$$-\frac{2}{\rho}\nabla p + \left(\frac{\mu_2+\eta}{\sqrt{\eta\omega}}\right)\varphi\hat{k} - \frac{\mu_1}{\sqrt{\omega}}w'\frac{\partial V}{\partial z}\hat{j} - \frac{\mu}{\sqrt{\omega}}v\frac{\partial V}{\partial z}\hat{k}$$

$$= -\frac{2}{\rho}\nabla^2(v\hat{j} + u'\hat{i} + w'\hat{k}) \quad \text{(A6)}$$

$$\frac{\mu_2+\eta}{\sqrt{\eta\omega}}w' = -\frac{2}{\rho}\nabla^2\varphi \quad \text{(A7)}$$

The natural boundary conditions at $z = 1$ are

$$\frac{2}{\rho}\frac{\partial\varphi}{\partial z} + \sqrt{\frac{\omega}{\eta}}\frac{\partial w'}{\partial z} = 0 \quad \text{(A8a)}$$

$$\frac{\partial v}{\partial z} = 0 \quad \text{(A8b)}$$

$$\sqrt{\frac{\omega}{\eta}}\frac{\partial\varphi}{\partial x} + \frac{2}{\rho}\frac{\partial u'}{\partial z} = 0 \quad \text{(A8c)}$$

The Fourier decomposition proceeds as before, except that $K_y = 0$ and $K_x = K$. Equations (2.6) and (2.7) become

$$(D^2 - K^2)^3 w' + K^2\left[\frac{(\mu_2+\eta)^2}{\omega\eta} + \frac{\mu_1^2}{\omega}\right]\frac{\rho^2}{4}w' = 0 \quad \text{(A9)}$$

$$K^2 p = (D^2 - K^2)\,D\,w' \quad \text{(A10)}$$

Equations (2.8b) and (2.9) become

$$\varphi = \sum_{i=1}^{6} A_i \frac{\rho}{2}\frac{\mu_2+\eta}{\sqrt{\omega\eta}}e^{im_iz} + A_7 e^{Kz} + A_8 e^{-Kz} \quad \text{(A11)}$$

$$(\mu_2+\eta)D\varphi = -\frac{2}{\rho}\sqrt{\eta\omega}\left[2D^3w' - K^2Dw' - \frac{1}{K^2}D^5w'\right] \quad \text{(A12)}$$

whereas (2.8a) remains the same. Instead of (2.10), Equations (A8b) and (A8c) become

$$D^2w' + \frac{\rho}{2}K^2\frac{\sqrt{\omega}}{\sqrt{\eta}}\varphi = 0 \quad \text{(A13)}$$

The solution technique for ρ is the same as before, only the boundary conditions to use are (A13), (A12), (2.11) and (A8a), and a slightly modified (2.4b):

$$\frac{\partial w'}{\partial z}(0) = w'(0) = \varphi(0) = w'(1) = 0 \quad \text{(A14)}$$

One point on the stability boundary, $Ra = 150$, $Re = 35$, was calculated using this technique. It was assumed that K and λ retain the same values as when $\omega = 1$. Although this was not verified the independence of λ_{opt} and K_{opt} was well established. The new stability point is at $Ma = 31.1$ (at $\omega = 2$ compared to $Ma = 22.2$ (at $\omega = 1$). While the stability boundary is raised significantly, note that it still falls below that for transverse disturbances.

Part III. Experiments

Convective instabilities in shear flow are studied experimentally in a horizontal concurrent two-phase flow channel. The convective instabilities are generated by interphase mass transfer between the two immiscible liquid phases. Schlieren photography perpendicular to the interface is used to record both the disturbance shape and its orientation. Three separate convective modes are studied: A. surface tension driven, B. buoyancy driven, and C. coactive buoyancy and surface tension driven. The disturbance structure of mode A appears as rings which oscillate in diameter as they are swept downstream. Streamwise oriented roll vortices also appear when flow conditions move the interface into a region of higher shear. Modes B and C are both characterized by streamwise oriented roll vortices with mode C having an additional fine structure.

SCOPE

Diffusive heat or mass transfer is often unstable with respect to convective instabilities. These instabilities can be from two possible sources. Most commonly, temperature or concentration variations cause differences in density. This may result in natural convection. Secondly, variations in temperature and concentration of an interface cause differences in surface tension and hence result in motion. While these two mechanisms cause many fascinating naturally occurring phenomena, they are also of great practical importance in industry. This importance arises from the manyfold enhancement of transport rates when convection is present.

Most interesting problems in this area, especially those of a practical sort, involve the presence of a primary shearing motion simultaneously occurring with heat or mass transfer. Unfortunately, most theoretical and experimental effort up until now has dealt with stagnant systems. In two previous papers (Gumerman and Homsy, 1974a, 1974b), we have dealt with theoretical aspects of the problem when shear is present. Specifically, criteria were developed to find if a system is stable or not. And, if the system is unstable, what orientation the disturbance will take.

This paper will deal with experiments concerning the effect of low Reynolds number shear on convective instabilities. The few previous experimental investigations in this area fall into two groups. In the first, enhancement